# On functions $F\colon \mathbb{F}_{2^{2t}} \to \mathbb{F}_{2^{2t}}$ mapping cosets of $\mathbb{F}_{2^t}^*$ to cosets of $\mathbb{F}_{2^t}^*$

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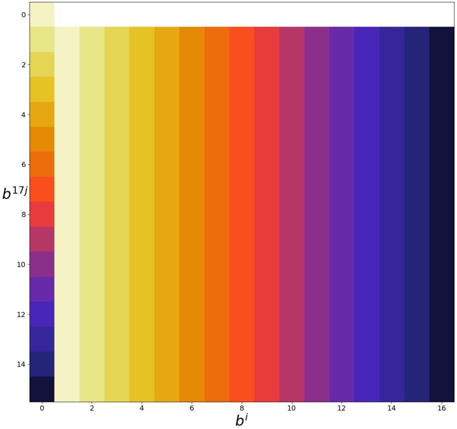
# Alignment




The Sbox $\pi$ [GOST standards Streebog/Kuznyechik]

- $\pi : \mathbb{F}_2^8 \to \mathbb{F}_2^8$ bijection specified as a look-up table
- Reversed-engineered [BirPerUdo16,PerUdo16,Per19]
- Happens to be extremely aligned !

# In this talk

1a) The Sbox $\pi$

1b) Bijections mapping $\gamma\mathbb{F}_{2^t}^*$ onto $G(\gamma)+\mathbb{F}_{2^t}^*$ (and their linearity)

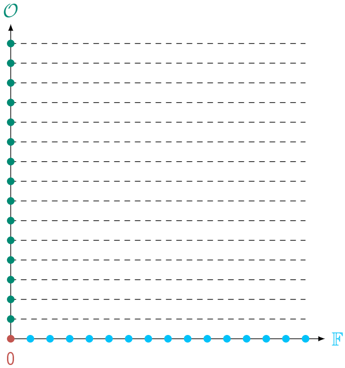
2a) The Kim mapping $\kappa$

2b) Functions mapping $\gamma\mathbb{F}_{2^t}^*$ onto $F(\gamma)\mathbb{F}_{2^t}^*$ (and their APN-ness)

# Partitions of $\mathbb{L}$ and $\mathbb{L}^*$

## Finite fields

- $\mathbb{F} \subset \mathbb{L}$ finite fields of characteristic 2.
- $\mathbb{F}$ additive subgroup of $\mathbb{L} \implies \mathbb{L} = \bigsqcup_{x \in \mathcal{O}} x + \mathbb{F}$



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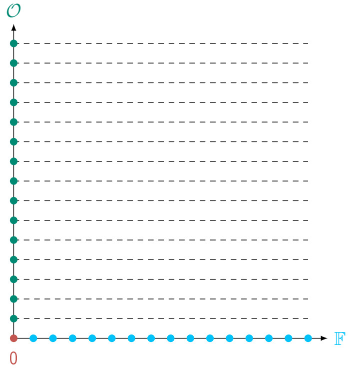
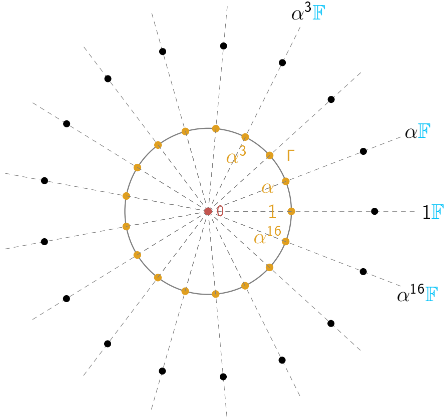
- $\mathbb{F}^*$ multiplicative subgroup of $\mathbb{L}^* \implies \mathbb{L}^* = \bigsqcup_{\gamma \in \Gamma} \gamma\mathbb{F}^*$



# Decomposition of $\pi$

$\mathbb{F}_2^8 \simeq \mathbb{L} = \mathbb{F}_{256}, \quad \mathbb{F} = \mathbb{F}_{16}$ $\lambda \in \mathbb{L}, \gamma \in \Gamma, \varphi \in \mathbb{F}$

16 additive cosets $x + \mathbb{F}, x \in \mathcal{O}$, 17 multiplicative cosets $\gamma\mathbb{F}^*, \gamma \in \Gamma$

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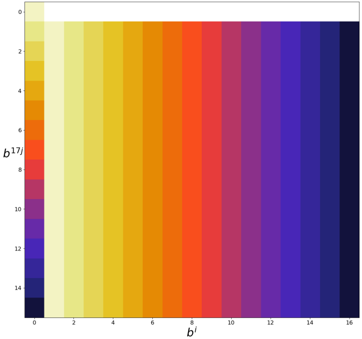
### Decomposition of $\pi$ [Perrin19]

With well-chosen $\mathbb{F}_2^8 \simeq \mathbb{L}$, $\Gamma$, $\mathcal{O}$, $\quad \pi$ can be expressed as:

$$\begin{aligned} \pi|_{\mathbb{L}\setminus\mathbb{F}} : \mathbb{L} \setminus \mathbb{F} &\to \mathbb{L} \\ \gamma\varphi &\mapsto G(\gamma) + F(\varphi) \end{aligned}$$

where $G : \Gamma \setminus \mathbb{F} \xrightarrow{\sim} \mathcal{O}$, $\quad F : \mathbb{F} \xrightarrow{\sim} \mathbb{F}$ $\quad$ with $F(0) = 0$ and $\pi(\mathbb{F}) = \mathcal{O}$.

$\pi(\gamma\varphi) = G(\gamma) + F(\varphi)$ when $\gamma \in \Gamma \setminus \mathbb{F}, \varphi \in \mathbb{F}^*$.

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## A few novelties

- The choice of $\mathcal{O}$ is understood.
- $G$ is understood.
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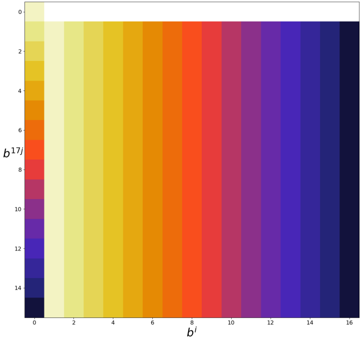


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**Can we say more about this structure ?**

# Generalizing $\pi$

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$[\mathbb{L} : \mathbb{F}] = 2 \qquad |\mathbb{L}^*| = 2^{2t} - 1 = (2^t - 1)(2^t + 1) \qquad |\Gamma| = 2^t + 1, \quad |\mathcal{O}| = |\mathbb{F}| = 2^t.$

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$$\widehat{\Pi}_\beta(\alpha) = \widehat{H}_{\varphi_\beta}(\mathrm{Tr}_{\mathbb{L}/\mathbb{F}}(\alpha)) - \widehat{H}_{\varphi_\beta}(0) + \sum_{\gamma\in\Gamma\setminus\mathbb{F}} (-1)^{\mathrm{Tr}_{\mathbb{L}/\mathbb{F}}(\beta G(\gamma))}\, \widehat{F}_{\mathrm{Tr}_{\mathbb{L}/\mathbb{F}}(\beta)}(\mathrm{Tr}_{\mathbb{L}/\mathbb{F}}(\alpha\gamma))$$

# Functions with low linearity

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$\Gamma = \mathbb{G}$, $F = \mathrm{Id}$. $\qquad$ For $\forall\ \alpha, \forall\ \beta \neq 0$, $\quad |\widehat{\Pi}_\beta(\alpha)| \leq 2^{t+2}$.

Best known bijections achieve $\leq 2^{t+1}$

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1a) The Sbox $\pi$ ☑

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# The enigmatic Kim function

**Kim mapping** [BDMW10]

$$\begin{aligned} \kappa : \quad \mathbb{F}_{64} &\to \mathbb{F}_{64} \\ x &\mapsto x^3 + x^{10} + ux^{24}; \end{aligned}$$

where $u$ is a specific root of $x^6 + x^4 + x^3 + x + 1$.

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**The reason of the fame**

- Optimal resistance against differential cryptanalysis (APN)
- Even number of variables and CCZ-equivalent to a bijection

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## Big APN problem

Up to CCZ-equivalence, does there exist any other APN permutation in even dimension ?

# The subspace property

A "special" property [BDMW10]

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**Mapping cosets onto cosets**

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$$\forall\lambda \in \mathbb{L},\ \exists\ G_\lambda : \mathbb{F} \to \mathbb{F} \text{ bijective s.t: } \quad \forall\varphi \in \mathbb{F}, \quad F(\lambda\varphi) = F(\lambda)G_\lambda(\varphi).$$

If $F(\lambda) \neq 0$, $G_\lambda$ unique

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**Cyclotomic mapping** [Wang07]

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- Also known as Wan Lidl polynomials [WanLidl91]
- Studies about graphs or permutations, [AkbWan07, BorPanWan23, Laigle-Chapuy07]
- only a few about cryptographic properties [ChenCoulter23, Gologlu23, BeiBriLea21]

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## Trivial relations

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# Subspace property and Cyclotomy

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## More trivial relations

- Gen. cyclotomic $\implies \forall\ \lambda,\quad F(\lambda\mathbb{F}) \subset F(\lambda)\mathbb{F}$.
- Gen. cyclotomic mapping satisfies the subspace property $\iff$ $\forall\ \lambda, \gcd(d_\lambda, |\mathbb{F}^*|) = 1$

# Spectral point of view (1/2)

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## Decomposition of Walsh coefficients

$\Gamma$ system of representatives, $\alpha, \beta \in \mathbb{L}$. $F : \mathbb{L} \to \mathbb{L}$ satisfying the subspace property. Then:

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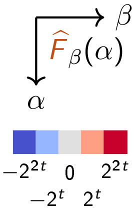
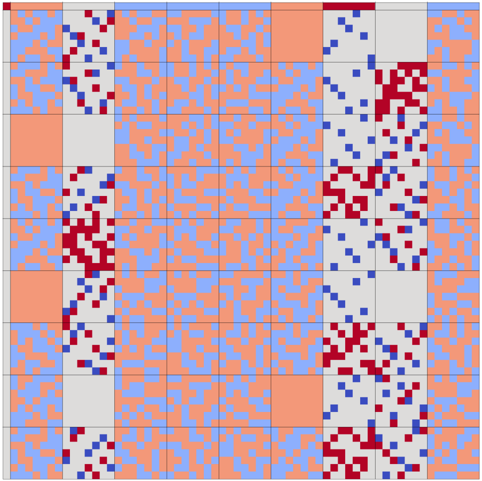
## Symmetries of Walsh coefficients

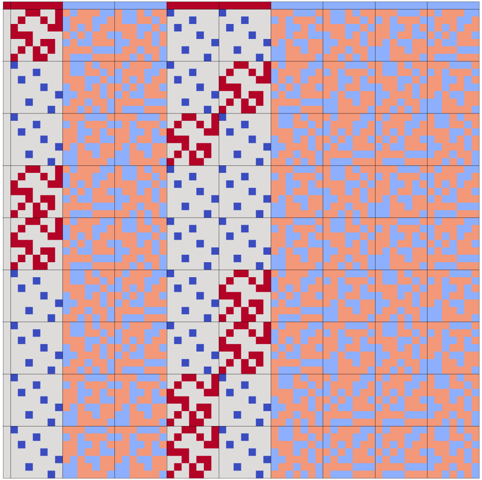
Let $G : \mathbb{F} \to \mathbb{F}$. $F$ satisfies the subspace property with $G_\lambda = G\ \forall\ \lambda$ if and only if:

$$\forall \alpha, \beta \in \mathbb{L},\ \forall \varphi \in \mathbb{F}^*, \quad \widehat{F}_{\beta G(\varphi)}(\alpha) = \widehat{F}_\beta(\alpha\varphi^{-1}).$$

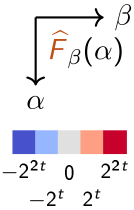


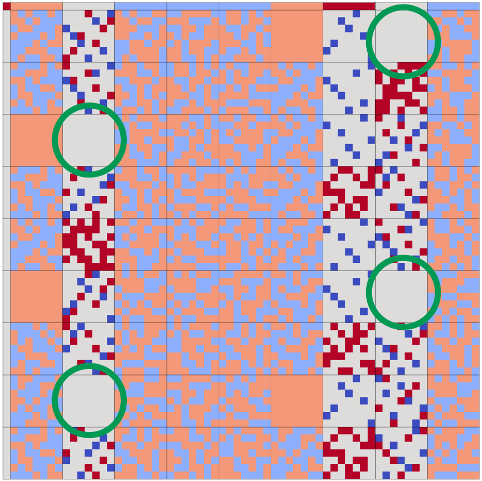



Kim mapping $\kappa: x \mapsto x^3 + x^{10} + ux^{24}$



Cube over $\mathbb{F}_{64}$ $x \mapsto x^3$



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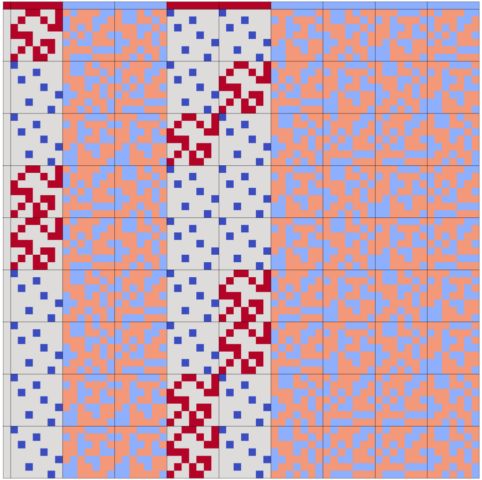

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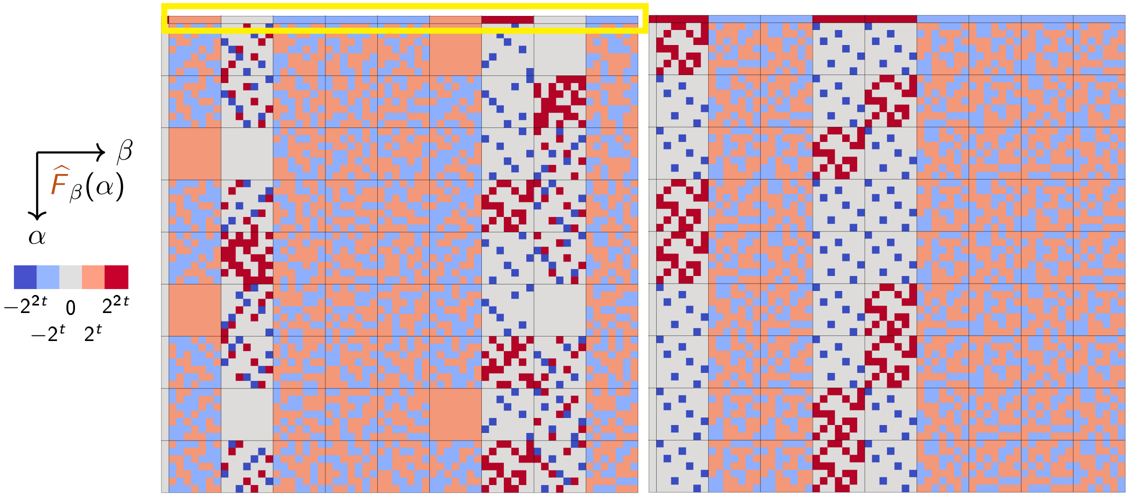


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# Walsh coefficients $\widehat{F}_\beta(0)$

*Subspace prop*: $\forall\ \lambda,\quad F(\lambda\mathbb{F}) = F(\lambda)\mathbb{F}.\qquad F(\lambda\varphi) = F(\lambda)G_\lambda(\varphi)$, with $G_\lambda : \mathbb{F} \xrightarrow{\sim} \mathbb{F}$

$$\widehat{F}_\beta(\alpha) := \sum_{\lambda\in\mathbb{L}} (-1)^{\mathrm{Tr}_{\mathbb{L}/\mathbb{F}_2}(\alpha\lambda+\beta F(\lambda))} \qquad N_\lambda := \frac{\left|F^{-1}(\lambda\mathbb{F})\right|}{|\mathbb{F}|}$$

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## Walsh coefficients in zero

$F$ satisfying the subspace property. $[\mathbb{L} : \mathbb{F}] = 2$. Then

$$\forall\ \beta \in \mathbb{L}^*, \quad \widehat{F}_\beta(0) = 2^t(N_{\beta^{-1}} - 1)$$

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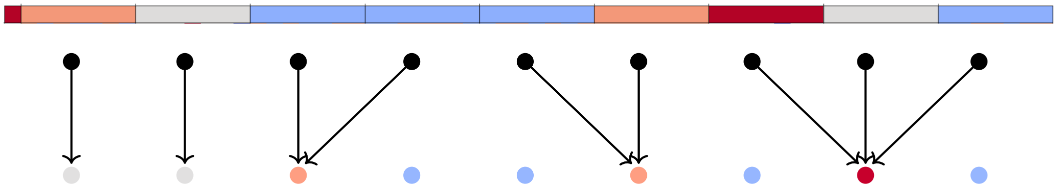

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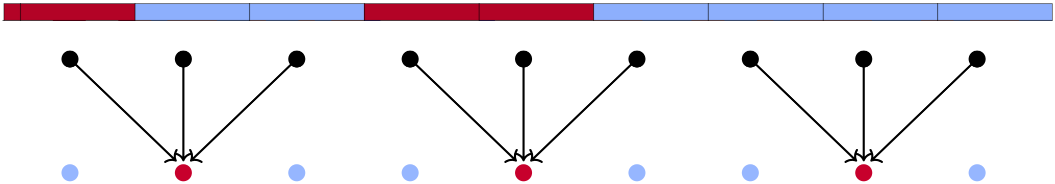

# Subspace property and APNness

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Subspace prop. when $[\mathbb{L} : \mathbb{F}] = 2 \implies \widehat{F}_\beta(0) = 2^t(N_{\beta^{-1}} - 1)$

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## Necessary condition to be APN

$F$ quadratic satisfying the subspace property. $[\mathbb{L} : \mathbb{F}] = 2$.

- If $F$ is APN then $\mathcal{N}_0 + \mathcal{N}_2 \geq \frac{2(2^t+1)}{3}$
- If $\mathcal{L}(F) = 2^{t+1}$ and $\mathcal{N}_0 + \mathcal{N}_2 \geq \frac{2(2^t+1)}{3}$ then $F$ is APN.

Proof: [BerCanChaLai06]

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## One already-solved case [Gologlu2023, ChaLis21]

$F$ quadratic cyclotomic when $[\mathbb{L} : \mathbb{F}] = 2$.

- If $t \neq 3$: $F$ APN $\iff$ $F \sim_{\text{CCZ}}$ Gold power
- If $t = 3$: $F$ APN $\iff$ $F \sim_{\text{CCZ}}$ Gold power or $F \sim_{\text{CCZ}} \kappa$.

# Squares of zeros

*Cyclotomic*: $\exists\ d, \forall\ \lambda, \forall\ \varphi, \quad F(\varphi\lambda) = \varphi^d F(\lambda)$ $\quad \mathbb{L} = \mathbb{F}_{2^{2t}}, \mathbb{F} = \mathbb{F}_{2^t}$

$$\widehat{F}_\beta(\alpha) := \sum_{\lambda \in \mathbb{L}} (-1)^{\mathrm{Tr}_{\mathbb{L}/\mathbb{F}_2}(\alpha\lambda + \beta F(\lambda))} \qquad Z_F := \left\{(\alpha, \beta), \widehat{F}_\beta(\alpha) = 0\right\} \cup \{(0,0)\}$$

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## Walsh zeroes

$F$ CCZ-equiv. to a bijection iff $\exists, U, V \subset Z_F$ subspaces of dim. $n$, $U \cap V = \{0\}$.

For $\kappa$, $U = u_1\mathbb{F} \times u_2\mathbb{F}$, $V = v_1\mathbb{F} \times v_2\mathbb{F}$.

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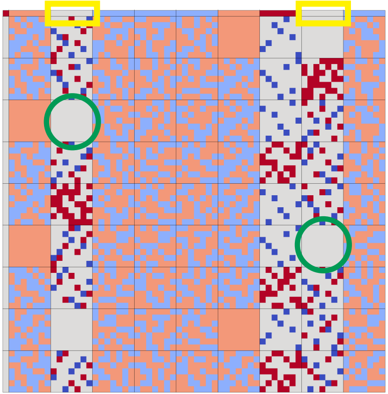

## Characterization of $\alpha\mathbb{F}^* \times \beta\mathbb{F}^* \subset Z_F$

For cyclotomic mappings of order $d$ over $\mathbb{F}$, $[\mathbb{L} : \mathbb{F}] = 2$

- $F(\mathbb{L}) = c\mathbb{F} \quad \implies \quad \forall\, \alpha,\ \alpha\mathbb{F}^* \times c\mathbb{F}^* \subset Z_F$
- Otherwise*, can only happened if $\widehat{F}_\beta(0) = 0$.

* Full characterization in the abstract

# Conclusion

## Generalizations of $\pi$ and linearity

Can we go below $\mathcal{L}(\Pi) \leq 2^{t+2}$ with other $\Gamma, \mathcal{O}, F, G$ ? constructions close to this one ?

## Subspace property / Cyclotomic mapping APNness

- Study of non-bijective cyclotomic mapping
- Still hope : non-quadratic or $[\mathbb{L} : \mathbb{F}] \neq 2 \ldots$
- Computer search

## CCZ-equivalence and bijectivity

Is the characterization of Walsh-zeros "square" sporadic or not ?

## Thanks ! ☺

# Subspace property and Cyclotomy

*Subspace prop*: $\forall\ \lambda$, $\quad F(\lambda\mathbb{F}) = F(\lambda)\mathbb{F}$

*Cyclotomic*: $\exists\ d, \forall\ \lambda, \forall\ \varphi$, $\quad F(\varphi\lambda) = \varphi^{d} F(\lambda)$

*Gen. cyclotomic*: $\forall\ \lambda, \exists\ d_\lambda, \forall\ \varphi$, $\quad F(\varphi\lambda) = \varphi^{d_\lambda} F(\lambda)$