A geometric construction of a class of non-linear MRD codes

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joint work with Nicola Durante and Giovanni Longobardi

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if equality holds

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$$\delta(\mathcal{C}) = \min\{\dim_{\mathbb{F}_q} \mathrm{Im}(\alpha - \beta)(y) : \alpha, \beta \in \mathcal{C}, \alpha \neq \beta\}$$

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$$\dim_{\mathbb{F}_q}(S \cap \Sigma) \leq \dim_{\mathbb{F}_{q^n}} S$$

U u-dim. $\mathbb{F}_q$-subspace of $V = \mathbb{F}_{q^n}^r$

$$L_U = \left\{ \langle \mathbf{u} \rangle_{\mathbb{F}_{q^n}} \in \mathrm{PG}(r-1, q^n) : \mathbf{u} \in U \setminus \{\mathbf{0}\} \right\}$$

$\mathbb{F}_q$-linear set of rank u

- $|L_U| \leq \frac{q^u-1}{q-1}$
- If $|L_U| = \frac{q^u-1}{q-1}$, then L_U **scattered** $\mathbb{F}_q$-linear set
- If $u = r$ and $\langle L_U \rangle = \mathrm{PG}(r-1, q^n)$, then L_U **canonical subgeometry** of $\mathrm{PG}(r-1, q^n)$

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where $x_j = x_{j,0} + x_{j,1}\xi + \ldots + x_{j,n-1}\xi^{n-1}$ for some $x_{j,i} \in \mathbb{F}_q$

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- Φ induces a map $\Phi' : \mathrm{PG}(m-1, q^n) \longrightarrow \mathrm{PG}(mn-1, q)$ sending $(h-1)$-dim. proj. subspaces to $(hn-1)$-dim. proj. subspaces

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$\mathcal{E} \subseteq \mathrm{PG}(r-1,q)$ **exterior set** with respect to $\Omega_h(\mathcal{A})$ if any line joining two distinct points of $\mathcal{E}$ is disjoint from $\Omega_h(\mathcal{A})$

Theorem (N. Durante, G.G.G., G. Longobardi)

Let $\mathcal{A} \subset \mathrm{PG}(r-1,q)$ *such that* $\langle \mathcal{A} \rangle = \mathrm{PG}(r-1,q)$. *Let* $\mathcal{E} \subset \mathrm{PG}(r-1,q)$ *be an exterior set with respect to* $\Omega_h(\mathcal{A})$, $0 \leq h \leq r-1$. *Then*

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$M, N \subseteq \mathrm{PG}(r-1,q)$ two disjoint sets with M a subspace

$\mathcal{K}(M,N) = \cup_{P \in M, Q \in N} PQ$

cone with vertex M **and base** N

Corollary (N. Durante, G.G.G., G. Longobardi)

Let $\mathcal{A} \subset \mathrm{PG}(r-1, q)$ *such that* $\langle \mathcal{A} \rangle = \mathrm{PG}(t-1, q)$, $1 \leq t < r$, *and let* $\mathcal{E} \subset \mathrm{PG}(r-1, q)$ *be an exterior set with respect to* $\Omega_h(\mathcal{A})$, $0 \leq h \leq r-1$. *Then* $\mathcal{E}$ *is contained in a cone* $\mathcal{K} = \mathcal{K}(S_{r-t-1}, \bar{\mathcal{E}})$, *with base* $\bar{\mathcal{E}} = \mathcal{E} \cap \langle \mathcal{A} \rangle$ *and vertex an* $(r-t-1)$*-dimensional subspace* S_{r-t-1} *complementary with* $\langle \mathcal{A} \rangle$. *Moreover,*

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$\mathcal{E}$ **maximum exterior set**

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Theorem

Let $\mathcal{E} \subseteq \mathrm{PG}(m-1, q^n)$ be an exterior set with respect to $\Omega_h(\Sigma_{m,n})$ and denote by $\mathcal{E}'$ the image of $\mathcal{E}$ under the field reduction. Then, the set

$$\mathcal{C} = \{\rho M : \langle M \rangle_{\mathbb{F}_q} \in \mathcal{E}', \rho \in \mathbb{F}_q\}$$

is an $(m, n, q; h+2)$-RD code closed under $\mathbb{F}_q$-multiplication. In addition, if $\mathcal{E}$ is maximum then $\mathcal{C}$ is MRD.

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Theorem (G. Donati, N. Durante - 2018)

Any C_F^σ-set is projectively equivalent to the set

$$\mathcal{X} = \{A, B\} \cup \bigcup_{a \in \mathbb{F}_q^*} \mathcal{X}_a,$$

$A = (0, \ldots, 0, 1)$, $B = (1, 0, \ldots, 0)$ *vertices of $\mathcal{X}$*

$\mathcal{X}_a = \{(1, t, t^{\sigma+1}, \ldots, t^{\sigma^{d-1}+\ldots+\sigma+1}) : \mathrm{N}_{q^n/q}(t) = a\}$ *components of $\mathcal{X}$*

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let $\Pi \cong \mathrm{PG}(d,q)$ be a subgeometry of $\mathcal{X}_1$

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For any $T \subseteq \mathbb{F}_q^*$, $1 \in T$, *the set*

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To the set $\mathcal{E}$ corresponds a $(d+1, n, q; d)$-MRD code $\mathcal{C}$, with $q > 2, n \geq 3$ and $2 \leq d \leq n-1$.

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projection of Σ from Λ^* to Λ

Definition (G. Lunardon - 2017)

$\Gamma = p_{\Lambda^*,\Lambda}(\Sigma)$ is an $(n-k+1)$-**embedding of** Σ if any $(n-k+1)$-subspace of Σ is disjoint from Λ^*.

Theorem (N. Durante, G.G.G., G. Longobardi)

- $\Sigma \cong \mathrm{PG}(n-1,q)$ *canonical subgeometry of* $\mathrm{PG}(n-1,q^n)$
- Λ^* *a* $(k-3)$*-subspace and* Λ *an* $(n-k+1)$*-subspace of* $\mathrm{PG}(n-1,q^n)$ *s.t.* $\Lambda^* \cap \Sigma = \emptyset = \Lambda^* \cap \Lambda^*$
- $\Gamma = p_{\Lambda^*,\Lambda}(\Sigma)$ $(n-k+1)$*-embedding of* Σ
- $\mathcal{E} \subseteq \Lambda$ *(maximum) exterior set with respect to* $\Omega_{n-k-1}(\Gamma)$

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If equality holds:

$\mathcal{K}(\Lambda^*,\mathcal{E})$ ext. set wrt $\Omega_{n-k-1}(\Sigma) \Longrightarrow \mathcal{E}$ ext. set wrt $\Omega_{n-k-1}(\Gamma)$

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and components

$$\mathcal{X}_a = \{(1, t, t^{\sigma+1}, \ldots, t^{\sigma^{n-k}+\ldots+\sigma+1}, 0, \ldots, 0) : \mathrm{N}_{q^n/q}(t) = a\}$$

For any $T \subseteq \mathbb{F}_q^*$, $1 \in T$, the set

$$\mathcal{E} = \left(\mathcal{X} \setminus \bigcup_{a \in T} \mathcal{X}_a \right) \cup \bigcup_{a \in T} J_a$$

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Corollary (N. Durante, G.G.G., G. Longobardi)

For any $T \subseteq \mathbb{F}_q^*$, $1 \in T$, the set $\mathcal{K} = \mathcal{K}(\Lambda^*, \mathcal{E})$ is a maximum exterior set with respect to $\Omega_{n-k-1}(\Sigma)$. Let $\mathcal{K}'$ be the image of $\mathcal{K}$ under the field reduction. Then the set

$$\mathcal{C}_{\sigma,T} = \{\rho M : \langle M \rangle_{\mathbb{F}_q} \in \mathcal{K}', \rho \in \mathbb{F}_q\}$$

is an $(n, n, q; d = n - k + 1)$-MRD code.

Known non-linear MRD codes:

- A. Cossidente, G. Marino, F. Pavese (2016)
- N. Durante, A. Siciliano (2018)
- G. Donati, N. Durante (2018): it is the punctured code $\mathcal{C}'_{\sigma,T} \subseteq \mathbb{F}_q^{(n-k+2)\times n}$ obtained from $\mathcal{C}_{\sigma,T}$ by deleting the last $(k-2)$ rows.
- K. Otal, F. Özbudak (2018)

The code $\mathcal{C}_{\sigma,T}$ in terms of σ-linearized polynomials is given by the union of the sets

$$\left\{\sum_{i=0}^{d} \lambda\alpha^{\sigma^i}\xi^{\frac{\sigma^i-1}{\sigma-1}}x^{\sigma^i} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \lambda,\alpha,\beta_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\xi) \in \mathbb{F}_q^* \setminus T\right\}$$

$$\left\{\lambda\alpha x + (-1)^{d+1}\lambda\alpha^{\sigma}\eta x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \lambda,\alpha,\beta_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\eta) \in T\right\}$$

$$\left\{\alpha x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha,\beta_i \in \mathbb{F}_{q^n}\right\} \cup \left\{\alpha x + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha,\beta_i \in \mathbb{F}_{q^n}\right\}$$

Let $1 \leq k \leq n$, the set

$$\mathcal{G}_{k,\sigma} = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} : \alpha_i \in \mathbb{F}_{q^n} \right\}$$

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Note that the sets

$$\mathcal{U} = \left\{ \alpha x^{\sigma^d} + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha, \beta_i \in \mathbb{F}_{q^n} \right\} = \left\{ f \circ x^{\sigma^d} : f \in \mathcal{G}_{k-1,\sigma} \right\}$$

$$\mathcal{V} = \left\{ \alpha x + \sum_{i=d+1}^{n-1} \beta_i x^{\sigma^i} : \alpha, \beta_i \in \mathbb{F}_{q^n} \right\} = \left\{ f \circ x^{\sigma^{d+1}} : f \in \mathcal{G}_{k-1,\sigma} \right\}$$

contained in $\mathcal{C}_{\sigma,T}$ are equivalent to $\mathcal{G}_{k-1,\sigma}$.

K. Otal and *F. Özbudak* (2018): let $I \subseteq \mathbb{F}_q$ and $1 \leq k \leq n-1$

$$\mathcal{C}^{(1)}_{n,k,\sigma,I} = \left\{ \sum_{i=0}^{k-1} \alpha_i x^{\sigma^i} : \alpha_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\alpha_0) \in I \right\}$$

$$\mathcal{C}^{(2)}_{n,k,\sigma,I} = \left\{ \sum_{i=1}^{k} \alpha_i x^{\sigma^i} : \alpha_i \in \mathbb{F}_{q^n}, \mathrm{N}_{q^n/q}(\alpha_k) \notin (-1)^{k(n+1)} I \right\}$$

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Corollary (K. Otal, F. Özbudak - 2018)

- *If $q = 2$ or $I \in \{\emptyset, \{0\}, \mathbb{F}_q^*, \mathbb{F}_q\}$, then $\mathcal{C}_{n,k,\sigma,I}$ is equivalent to a generalized Gabidulin code $\mathcal{G}_{k,\sigma}$*
- *If $q > 2$ and $I \notin \{\emptyset, \{0\}, \mathbb{F}_q^*, \mathbb{F}_q\}$, then $\mathcal{C}_{n,k,\sigma,I}$ is a non-linear code*

Theorem (N. Durante, G.G.G., G. Longobardi)

If $q = 2$ or $T = \mathbb{F}_q^*$ and $I \in \{\{0\}, \mathbb{F}_q, \mathbb{F}_q^*, \emptyset\}$, then the codes of type $\mathcal{C}_{n,k,\sigma,I}$ and $\mathcal{C}_{\sigma,T}$ are both equivalent to a $\mathcal{G}_{k,\sigma}$.

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The code $\mathcal{C}_{n,k,\sigma,I}$ contains the set

$$\mathcal{W} = \left\{ \sum_{i=1}^{k-1} \gamma_i x^{\sigma^i} : \gamma_i \in \mathbb{F}_{q^n} \right\} = \{ f \circ x^{\sigma} : f \in \mathcal{G}_{k-1,\sigma} \}$$

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Let $I \notin \{\{0\}, \mathbb{F}_q, \mathbb{F}_q^*, \emptyset\}$, $1 \in T \subseteq \mathbb{F}_q^*$. Then the codes of type $\mathcal{C}_{n,k,\sigma,I}$ and $\mathcal{C}_{\sigma,T}$ are neither equivalent nor adjointly equivalent.

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- Investigating a geometric analogue via exterior sets for constructing new sum-rank metric codes.

Thanks for your attention!