Further results on orbits and incidence matrices for the class O_6 of lines external to the twisted cubic in $\mathrm{PG}(3, q)$

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joint work with
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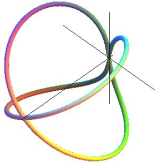
OUTLINE

1. The twisted cubic in $\mathrm{PG}(3;q)$ and its stabilizer group G_q
2. Orbits of points, planes, lines
3. Classes of lines
4. Orbits of lines for all classes but $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$
5. Orbits of lines for class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$

The twisted cubic in $\mathrm{PG}(3;q)$

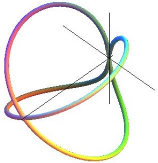


$$\mathscr{C} = \{P(t) = \mathbf{P}(t^3, t^2, t, 1) \mid t \in \mathbb{F}_q\} \cup \mathbf{P}(1,0,0,0)$$

G_q: stabilizer group of $\mathscr{C}$, $\quad G_q \cong \mathrm{PGL}(2,q)$

$$\mathbf{M} = \begin{bmatrix} a^3 & a^2c & ac^2 & c^3 \\ 3a^2b & a^2d + 2abc & bc^2 + 2acd & 3c^2d \\ 3ab^2 & b^2c + 2abd & ad^2 + 2bcd & 3cd^2 \\ b^3 & b^2d & bd^2 & d^3 \end{bmatrix}, \quad ad - bc \neq 0.$$

The twisted cubic in $\mathrm{PG}(3;q)$



$$\mathscr{C} = \{P(t) = \mathbf{P}(t^3, t^2, t, 1) \mid t \in \mathbb{F}_q\} \cup \mathbf{P}(1,0,0,0)$$

G_q: stabilizer group of $\mathscr{C}$

- orbits of points
- orbits of planes
- orbits of lines ?

The twisted cubic in $\mathrm{PG}(3;q)$

D. Bartoli, A.A. Davydov, F. Pambianco, S.M.
On planes through points off the twisted cubic in PG(3,q) and multiple covering codes,
Finite Fields Appl. **67**, (2020)

Point-plane incidence matrix

$\Downarrow$

Optimal multiple covering codes

The twisted cubic in $\mathrm{PG}(3;q)$

Point-plane incidence matrix

$\Downarrow$

A.A. Davydov, F. Pambianco, S.M.
On cosets weight distributions of the doubly-extended Reed-Solomon codes of codimension 4,
IEEE Trans. Inform. Theory, **67**(8), (2021)

Classification of the cosets of the $[q+1, q-3, 5]_q3$ generalized doubly-extended Reed-Solomon code

What about the orbits of lines of G_q?

Theorem [Hirschfeld book, Chapter 21] *The lines of* $\mathrm{PG}(3,q)$ *can be partitioned into classes called* $\mathcal{O}_i$ *and* $\mathcal{O}'_i$, $\mathcal{O}'_i = \mathcal{O}_i\mathfrak{A}$, *each of which is a union of orbits under* G_q.

What about the orbits of lines of G_q?

$\diamond$ $q \not\equiv 0 \pmod 3$, $q \geq 5$, $\mathcal{O}'_i = \mathcal{O}_i\mathfrak{A}$, $\#\mathcal{O}'_i = \#\mathcal{O}_i$, $i = 1, \ldots, 6$.

- $\mathcal{O}_1 = \mathcal{O}_{\mathrm{RC}} = \{\mathrm{RC}\text{-lines}\}$, $\mathcal{O}'_1 = \mathcal{O}_{\mathrm{RA}} = \{\mathrm{RA}\text{-lines}\}$, $\#\mathcal{O}_{\mathrm{RC}} = \#\mathcal{O}_{\mathrm{RA}} = (q^2+q)/2$;
- $\mathcal{O}_2 = \mathcal{O}'_2 = \mathcal{O}_{\mathrm{T}} = \{\mathrm{T}\text{-lines}\}$, $\#\mathcal{O}_{\mathrm{T}} = q+1$;
- $\mathcal{O}_3 = \mathcal{O}_{\mathrm{IC}} = \{\mathrm{IC}\text{-lines}\}$, $\mathcal{O}'_3 = \mathcal{O}_{\mathrm{IA}} = \{\mathrm{IA}\text{-lines}\}$, $\#\mathcal{O}_{\mathrm{IC}} = \#\mathcal{O}_{\mathrm{IA}} = (q^2-q)/2$;
- $\mathcal{O}_4 = \mathcal{O}'_4 = \mathcal{O}_{\mathrm{U}\Gamma} = \{\mathrm{U}\Gamma\text{-lines}\}$, $\#\mathcal{O}_{\mathrm{U}\Gamma} = q^2+q$;
- $\mathcal{O}_5 = \mathcal{O}_{\mathrm{Un}\Gamma} = \{\mathrm{Un}\Gamma\text{-lines}\}$, $\mathcal{O}'_5 = \mathcal{O}_{\mathrm{E}\Gamma} = \{\mathrm{E}\Gamma\text{-lines}\}$, $\#\mathcal{O}_{\mathrm{Un}\Gamma} = \#\mathcal{O}_{\mathrm{E}\Gamma} = q^3-q$;
- $\mathcal{O}_6 = \mathcal{O}'_6 = \mathcal{O}_{\mathrm{En}\Gamma} = \{\mathrm{En}\Gamma\text{-lines}\}$, $\#\mathcal{O}_{\mathrm{En}\Gamma} = (q^2-q)(q^2-1)$.

What about the orbits of lines of G_q?

$\diamond$ $q \equiv 0 \pmod 3$, $q > 3$.

- Classes $\mathcal{O}_1, \ldots, \mathcal{O}_6$ are as in the previous slide;
- $\mathcal{O}_7 = \mathcal{O}_{\mathrm{A}} = \{\text{A-line}\}$, $\#\mathcal{O}_{\mathrm{A}} = 1$;
- $\mathcal{O}_8 = \mathcal{O}_{\mathrm{EA}} = \{\text{EA-lines}\}$, $\#\mathcal{O}_{\mathrm{EA}} = (q+1)(q^2-1)$.

What about the orbits of lines of G_q?

A.A. Davydov, F. Pambianco, S.M.
Orbits of lines for a twisted cubic in $\mathrm{PG}(3, q)$,
Mediterr. J. Math. **20**(3), (2023)
arXiv:2103.12655 (2021)

We determine the orbits of lines for all classes of lines except
$\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma} = \{\mathrm{En}\Gamma\text{-lines}\}$

The classes contain 1 or 2 or 3 orbits

The orbits of lines of G_q

A.A. Davydov, F. Pambianco, S.M.
Twisted cubic and point-line incidence matrix in $\mathrm{PG}(3,q)$,
Des. Codes Cryptogr. **89**(10), (2021)

Point-line incidence matrix

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Configurations

$\Downarrow$

Bipartite graph codes free of 4-cycles

The orbits of lines of G_q

A.A. Davydov, F. Pambianco, S.M.
Twisted cubic and plane-line incidence matrix in $\mathrm{PG}(3,q)$,
J. Geom. **113**(2), (2022)
arXiv:2103.12655 (2021)

Plane-line incidence matrix

$\Downarrow$

Configurations that not contains
2 x 2 sub-matrices whose entries are 1

$\Downarrow$

Zarankiewicz problem
Bipartite graph codes free of 4-cycles

The orbits of lines of G_q

Related works

A. Blokhuis, R. Pellikaan, T. Szönyi
The extended coset leader weight enumerator of a twisted cubic code,
Des. Codes Cryptogr. **90**, (2022)
arXiv:2103.16904 (2021)

Orbits of lines for all classes of lines except $\mathcal{O}_6$,
$q \geq 23$

The orbits of lines of G_q

Related works

G. Günay, M. Lavrauw
On pencils of cubics on the projective line over finite fields of characteristic > 3,
Finite Fields Appl. **78**, (2022)
arXiv:2104.04756 (2021)

Orbits of lines for all classes of lines except $\mathcal{O}_6$,
odd $q, q \not\equiv 0 \pmod 3$

They also determine the number of distinct planes through distinct lines and distinct points through distinct lines

What about the orbits of lines of class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$?

A.A. Davydov, F. Pambianco, S.M.
Orbits of lines for a twisted cubic in $\mathrm{PG}(3,q)$,
Mediterr. J. Math. **20**(3), (2023)
arXiv:2103.12655 (2021)

Computer search for odd $q \le 37$, odd $q \le 37$, even $q \le 64$

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Conjecture

What about the orbits of lines of class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$?

Conjecture

Let q be even,

$q \equiv \xi \pmod 3$, $\xi \in \{1, -1\}$.

The total number of $\mathrm{En}\Gamma$*-line orbits is* $2q - 2 + \xi$.

$2 + \xi$ *orbits of length* $(q^3 - q)/(2 + \xi)$;
$2q - 4$ *orbits of length* $(q^3 - q)/2$.

What about the orbits of lines of class $\mathscr{O}_6 = \mathscr{O}_{\mathrm{En}\Gamma}$?

Conjecture

Let q be odd, $q \equiv \xi \pmod 3$, $\xi \in \{1, -1, 0\}$.

The total number of $\mathrm{En}\Gamma$*-line orbits is* $2q - 3 + \xi$.

$n_q(\xi)$	*orbits of length*	$(q^3 - q)/4$,
$q - 1$	*orbits of length*	$(q^3 - q)/2$,
$(q - \xi)/3$	*orbits of length*	$q^3 - q$,

where

$$n_q(1) = (2q - 11)/3,$$
$$n_q(-1) = (2q - 10)/3,$$
$$n_q(0) = (2q - 6)/3.$$

In addition, for $q \equiv 1 \pmod 3$*, there are:*

1	*orbit of length*	$(q^3 - q)/12$,
2	*orbits of length*	$(q^3 - q)/3$.

What about the orbits of lines of class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$?

A.A. Davydov, F. Pambianco, S.M.
Orbits of the class $\mathcal{O}_6$ of lines external to the twisted cubic in $\mathrm{PG}(3, q)$,
Mediterr. J. Math. 20(3), (2023)
arXiv: 2209.04910 (2022)

What about the orbits of lines of class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$?

$$\mathcal{L} = \overline{\mathbf{P}(1,0,0,1)\mathbf{P}(0,0,1,0)}$$

Theorem

Let $q \not\equiv 0 \pmod 3$, $q \equiv \xi \pmod 3$.
Let $G_q^{\mathcal{L}}$ be the subgroup of G_q fixing $\mathcal{L}$.
Let $\mathscr{O}_{\mathcal{L}}$ be the orbit of $\mathcal{L}$ under G_q.

(*i*) $\xi = 1$, q *is even or* $-1/2$ *is a non-cube in* $\mathbb{F}_q$,
$\#G_q^{\mathcal{L}} = 3$, $\#\mathscr{O}_{\mathcal{L}} = 1/3(q^3 - q)$.

(*ii*) $\xi = 1$, q *is odd and* $-1/2$ *is a cube in* $\mathbb{F}_q$,
$\#G_q^{\mathcal{L}} = 12$, $\#\mathscr{O}_{\mathcal{L}} = 1/12(q^3 - q)$, $\#G_q^{\mathcal{L}} \cong A_4$.

(*iii*) $\xi = -1$, q *is even*, $\#G_q^{\mathcal{L}} = 1$, $\#\mathscr{O}_{\mathcal{L}} = q^3 - q$.

(*iv*) $\xi = -1$, q *is odd*, $\#G_q^{\mathcal{L}} = 2$, $\#\mathscr{O}_{\mathcal{L}} = 1/2(q^3 - q)$.

The $q-2$ distinct orbits of ℓ_μ-lines, $\mu \in \mathbb{F}_q \setminus \{0,1\}$, for even q

$$\ell_\mu = \overline{\mathbf{P}(0,\mu,0,1)\mathbf{P}(1,0,1,0)}, \mu \in \mathbb{F}_q \setminus \{0,1\}$$

Theorem

Let q be even, $q \geq 8$.

Any two lines $\ell_{\mu'}$, $\ell_{\mu''}$ belong to different orbits of G_q.
No ℓ_μ-line belongs to the orbit $\mathscr{O}_{\mathcal{L}}$ of the line $\mathcal{L}$.

$\#G_q^{\ell_\mu} = 2.$
$\#\mathscr{O}_\mu = (q^3 - q)/2.$

The orbits of lines ℓ_μ, $\mu \in \mathbb{F}_q \setminus \{0,1\}$, $q \equiv 0 \pmod 3$

$$\ell_\mu = \overline{\mathbf{P}(0,\mu,0,1)\mathbf{P}(1,0,1,0)}, \mu \in \mathbb{F}_q \setminus \{0,1\}$$

Theorem

Let $q \equiv 0 \pmod 3$, $q \geq 9$.

If $\mu \in \not\square$, all lines ℓ_μ belong to distinct orbits of G_q.
$\#G_q^{\ell_\mu} = 2$, $\quad \#\mathscr{O}_\mu = (q^3 - q)/2$.

If $\mu \in \square$, ℓ_μ and $\ell_{\mu'}$ belong to the same orbit $\iff$
$\mu = d^4$, $\mu' = d^4 + d^2 + 1$, $1 - d^2 \in \square$, $d \in \mathbb{F}_q \setminus \{0, \pm 1\}$,
and also $d \neq \pm\sqrt{-1}$ if $q \equiv 1 \pmod 4$.

$q \equiv -1 \pmod 4 \implies$ *at most 2 ℓ_μ-lines belong to the same orbit;*
$q \equiv 1 \pmod 4 \implies$ *at most 3 ℓ_μ-lines belong to the same orbit;*
$G_q^{\ell_\mu} \cong C_2 \times C_2$, $\#\mathscr{O}_\mu = (q^3 - q)/4$.

The orbits of lines ℓ_μ, $\mu \in \mathbb{F}_q \setminus \{0, 1, 1/9\}$, q odd, $q \not\equiv 0 \pmod 3$

$$\ell_\mu = \overline{\mathbf{P}(0,\mu,0,1)\mathbf{P}(1,0,1,0)}, \mu \in \mathbb{F}_q \setminus \{0,1\}$$

Theorem

The line $\mathcal{L}$ and a line ℓ_μ belong to the same orbit $\iff$

- $q \equiv -1 \pmod{12}$, $\mu = -1/3$, $\mu \in \not\square$
- $q \equiv 1 \pmod{12}$, $\mu \in \square$, $1/2$ *is a cube*, $-1/3$ *is a fourth power.*

The orbits of lines ℓ_μ, $\mu \in \mathbb{F}_q \setminus \{0, 1, 1/9\}$, q odd, $q \not\equiv 0 \pmod 3$

$$\ell_\mu = \overline{\mathbf{P}(0,\mu,0,1)\mathbf{P}(1,0,1,0)},\ \ \mu \in \mathbb{F}_q \setminus \{0,1\}$$

Theorem

- $\mu \in \not\square \Longrightarrow \ \#G_q^{\ell_\mu} = 2$, $\#\mathscr{O}_\mu = (q^3 - q)/2$
- $\mu \in \square$ and *either $\mu \neq -1/3$, or $q \not\equiv 1 \pmod{12}$, or $-1/3$ is not a fourth power* $\Longrightarrow \ G_q^{\ell_\mu} \cong C_2 \times C_2$, $\#\mathscr{O}_\mu = (q^3 - q)/4$
- $\mu = -1/3, \ \Longrightarrow \ \mu \in \square$, $G_q^{\ell_\mu} \cong A_4$, $\#\mathscr{O}_\mu = (q^3 - q)/12.$

The lines ℓ_μ and $\mathcal{L}$

sketch of the proof

- determine the action of the group G_q on the lines of the family ℓ_μ and the line $\mathcal{L}$
- determine the stabilizer group of the lines ℓ_μ and $\mathcal{L}$ under the group G_q
- investigate when two lines belong to the same orbit

What about the orbits of lines of class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$?

A.A. Davydov, F. Pambianco, S.M.
Incidence matrices for the class $\mathcal{O}_6$ of lines external to the twisted cubic in $\mathrm{PG}(3, q)$,
J. Geom. 114(2), (2023)

Point-line incidence matrix
Plane-line incidence matrix

$\Downarrow$

Configurations that not contains
2 x 2 sub-matrices whose entries are 1

$\Downarrow$

Zarankiewicz problem
Bipartite graph codes free of 4-cycles

What about the orbits of lines of class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$?

Conjecture

Let q be even, $q \equiv \xi \pmod 3$, $\xi \in \{1, -1\}$.

The total number of $\mathrm{En}\Gamma$-line orbits is $2q - 2 + \xi$.

$2 + \xi$	*orbits of length*	$(q^3 - q)/(2 + \xi)$;
$2q - 4$	*orbits of length*	$(q^3 - q)/2$.

proved by

M. Ceria, F. Pavese, *On the geometry of a $(q + 1)$-arc of $\mathrm{PG}(3, q)$, q even*, Discrete Math. **346**, (2023)

They consider the Plücker correspondence, which sends the lines of PG(3, q) to the points of the Klein quadric $Q^+(5, q)$

They also determine the point-line incidence matrix

What about the orbits of lines of class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$?

A.A. Davydov, F. Pambianco, S.M.
Further Results on Orbits and Incidence matrices for the Class $\mathcal{O}_6$ of Lines External to the Twisted Cubic in $\mathrm{PG}(3,q)$
WCC 2024

A family of $\mathscr{L}_\rho$-lines, $\rho \in \mathbb{F}_q \setminus \{0\}$, $q \not\equiv 0 \pmod 3$

$$\mathscr{L}_\rho = \overline{\mathbf{P}(\rho,0,0,1)\mathbf{P}(0,0,1,0)},\ \rho \in \mathbb{F}_q \setminus \{0\},\ q \not\equiv 0 \pmod 3$$

Lemma

- $\mathscr{L}_0 = \mathcal{T}_0$.
- $\mathscr{L}_1 = \mathscr{L}$.
- *For* $q \not\equiv 0 \pmod 3$, *the line* $\mathscr{L}_\rho$, $\rho \neq 0$, *is an* $\mathrm{En}\Gamma$*-line.*
- *For* $q \equiv 0 \pmod 3$, *the line* $\mathscr{L}_\rho$ *is not an* $\mathrm{En}\Gamma$*-line.*

Stabilizer and orbits of $\mathscr{L}_\rho$-lines

Theorem

- *Let* $q \equiv 1 \pmod 3$*. Let* q *be even or let* -2ρ *be a non-cube in* $\mathbb{F}_q$*.*
  Then $\#G_q^{\mathscr{L}_\rho} = 3; \#\mathscr{O}_\rho = (q^3 - q)/3.$
- *Let* $q \equiv 1 \pmod 3$*. Let* q *be odd and let* -2ρ *be a cube in* $\mathbb{F}_q$*.*
  Then $\#G_q^{\mathscr{L}_\rho} = 12$ *and* $G_q^{\mathscr{L}_\rho} \cong \mathbf{A}_4; \#\mathscr{O}_\rho = (q^3 - q)/12.$
- *Let* $q \equiv -1 \pmod 3$*. Let* q *be even.*
  Then $\#G_q^{\mathscr{L}_\rho} = 1$ *and* $\#\mathscr{O}_\rho = q^3 - q.$
- *Let* $q \equiv -1 \pmod 3$*. Let* q *be odd.*
  Then $\#G_q^{\mathscr{L}_\rho} = 2$ *and* $\#\mathscr{O}_\rho = (q^3 - q)/2.$

Does the family $\mathscr{L}_\rho$-lines give us new orbits?

Theorem

- *Let q be even or*
- *let q be odd, $q \not\equiv 0 \pmod 3$, and -2ρ be a non-cube in $\mathbb{F}_q$.*

Then every orbit $\mathbb{O}_\rho$ is different from any orbit $\mathscr{O}_\mu$.

sketch of the proof

- consider intersections of $\mathscr{L}_\rho$-lines and tangents
- $\mathscr{L}_\rho$ and $\mathcal{T}_t$ intersect $\iff \varpi(\mathscr{L}_\rho, \mathcal{T}_t) = t^4 + 2\rho t = 0$
- Let $\mathfrak{n}_q(\cdot)$ be the number of T -points on a line
- $\mathfrak{n}_q(\rho) \neq \mathfrak{n}_q(\mu) \implies$ the orbits $\mathbb{O}_\rho$ and $\mathscr{O}_\mu$ are distinct

Does the family $\mathscr{L}_\rho$-lines give us new orbits?

Theorem

Let q be even. Then $\mathbb{O}_\rho \neq \mathscr{O}_\mu$. Moreover:

- *Let $q = 2^{2m}$.*
  $\mathscr{L}_{\rho'}$ and $\mathscr{L}_{\rho''}$ belong to different orbits
  $\iff \log\rho' \not\equiv \log\rho'' \pmod 3$.

  Let α be a primitive element of $\mathbb{F}_q$.
  $\mathscr{L} = \mathscr{L}_1, \mathscr{L}_\alpha, \mathscr{L}_{\alpha^2}$ generate the 3 distinct $\frac{1}{3}(q^3-q)$-orbits.

- *Let $q = 2^{2m-1} \equiv -1 \pmod 3$*
  all $\mathscr{L}_\rho$-lines generate the same (q^3-q)-orbit.

Does the family $\mathscr{L}_\rho$-lines give us new orbits?

Theorem

Let q be odd, $q \equiv -1 \pmod 3$.

Then all $\mathscr{L}_\rho$-lines generate the same $\frac{1}{2}(q^3 - q)$-orbit $\mathbb{O}_{\mathcal{L}_1}$.

It is different from any orbit $\mathscr{O}_\mu$ except when $q \equiv -1 \pmod{12}$; in this case $\mathbb{O}_1 = \mathscr{O}_{-1/3}$ generated by the line $\ell_{-1/3}$.

Does the family $\mathscr{L}_\rho$-lines give us new orbits?

Theorem

Let q be odd, $q \equiv 1 \pmod 3$.

- *$\mathscr{L}_{\rho'}$ and $\mathscr{L}_{\rho''}$ belong to different orbits $\iff \log \rho' \not\equiv \log \rho'' \pmod 3$.*
- *$\mathbb{O}_\rho^{(1)}$ and $\mathbb{O}_\rho^{(2)}$ have size $\frac{1}{3}(q^3 - q)$ and are generated by lines $\mathscr{L}_\rho$ such that -2ρ is a non-cube in $\mathbb{F}_q$. They are different from any orbit $\mathscr{O}_\mu$.*
- *$\mathbb{O}_\rho^{(3)}$ has size $\frac{1}{12}(q^3 - q)$ and is generated by a line $\mathscr{L}_\rho$ such that -2ρ is a cube in $\mathbb{F}_q$. If $q \not\equiv 1 \pmod{12}$ or $-1/3$ is not a fourth degree in $\mathbb{F}_q$, it is different from any orbit $\mathscr{O}_\mu$. Otherwise $\mathbb{O}_\rho^{(3)} = \mathscr{O}_{-1/3}$.*

Does the family $\mathscr{L}_\rho$-lines give us new orbits?

sketch of the proof

Let $\mathfrak{R}_m \triangleq \{\rho \in \mathbb{F}_q^* \mid \log \rho \equiv m \pmod 3\}$

- $\log \rho_1 \equiv \log \rho_2 \pmod 3 \implies \mathscr{L}_{\rho_1}, \mathscr{L}_{\rho_2}$ belong to the same orbit

$\log(\rho_1/\rho_2) = \log \rho_1 - \log \rho_2 \implies \rho_1/\rho_2$ is a cube. Let $d^3 = \rho_1/\rho_2$

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d^2 & 0 \\ 0 & 0 & 0 & d^3 \end{bmatrix}$$

$$\mathbf{P}[0,0,1,0] \times \mathbf{M} = \mathbf{P}[0,0,d^2,0] = \mathbf{P}[0,0,1,0]$$

$$\mathbf{P}[\rho_1,0,0,1] \times \mathbf{M} = \mathbf{P}[\rho_1,0,0,d^3] = \mathbf{P}[1,0,0,d^3/\rho_1] = \mathbf{P}[1,0,0,1/\rho_2]$$
$$= \mathbf{P}[\rho_2,0,0,1]$$

Does the family $\mathscr{L}_\rho$-lines give us new orbits?

- $\rho_1 \in \mathfrak{R}_{m_1}, \rho_2 \in \mathfrak{R}_{m_2}, -2\rho_1, -2\rho_2$ are non-cubes $\Longrightarrow$

$\mathscr{L}_{\rho_1}, \mathscr{L}_{\rho_2}$ generate two distinct orbits

$[0,0,1,0] \times \mathbf{M}^{\Psi} = [3ab^2, b^2c + 2abd, ad^2 + 2bcd, 3cd^2] = [\rho_2, 0, \gamma, 1]$

This implies $ab^2/\rho_2 = cd^2$ and $a, b, c, d \neq 0$. Put $b = 1$

$a/\rho_2 = cd^2, c + 2ad = 0 \Longrightarrow$

$d^3 = -1/2\rho_2$, contradiction as $-1/2\rho_2$ (as $-2\rho_2$) is not a cube

Does the family $\mathscr{L}_\rho$-lines give us new orbits?

$$[0,0,1,0] \times \mathbf{M}^\infty = [0,0,\gamma,0]$$

$$\mathbf{M}^\infty = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d^2 & 0 \\ 0 & 0 & 0 & d^3 \end{bmatrix}$$

$$[\rho_1, 0, 0, 1]\mathbf{M}^\infty = [\rho_2, 0, \gamma, 1]$$

$$[\rho_1, 0, 0, 1] \times \mathbf{M}^\infty = [\rho_1, 0, 0, d^3] \implies d^3 = \rho_1/\rho_2$$

$$m_1 \neq m_2 \implies \log(\rho_1/\rho_2) \not\equiv 0 \pmod 3$$

therefore ρ_1/ρ_2 is not a cube, contradiction

Thus, a projectivity $\Psi \in G_q$ sending $\mathscr{L}_{\rho_1}$ to $\mathscr{L}_{\rho_2}$ does not exist

Point-line and plane line matrix incidences for $\mathscr{L}_\rho$-lines

Theorem

Let q be even

$$\widetilde{\mathbb{W}}_q(\rho) \triangleq \#\left\{\gamma \mid \mathrm{Tr}_2\left(\frac{\gamma^3}{\rho}+1\right)=1,\ \gamma\in\mathbb{F}_q,\ q=2^c,\ \rho\in\mathbb{F}_q^* \text{ is fixed}\right\}$$

For the point-line incidence matrix corresponding to the orbit $\mathbb{O}_\rho$ of a line $\mathscr{L}_\rho$ the following holds:

- *Let $q = 2^{2m-1}$. Then $\#\mathbb{O}_\rho = q^3 - q$ for all ρ; $\widetilde{\mathbb{W}}_q(\rho) = q/2$, and*

$$\mathbb{P}_{\mathrm{T}} = 1, \mathbb{L}_{\mathrm{T}} = q-1;\ 2\mathbb{P}_{1_\Gamma} = \mathbb{L}_{1_\Gamma} = q;\ 6\mathbb{P}_{3_\Gamma} = \mathbb{L}_{3_\Gamma} = q-2;$$
$$3\mathbb{P}_{0_\Gamma} = \mathbb{L}_{0_\Gamma} = q+1.$$

Point-line and plane line matrix incidences for $\mathscr{L}_\rho$-lines

- Let $q = 2^{2m}$. Then $\#\mathbb{O}_\rho = \frac{1}{3}(q^3 - q)$ for all ρ and

$$\mathbb{P}_{\mathrm{T}} = 1,\ \mathbb{L}_{\mathrm{T}} = \frac{1}{3}(q-1);\ \mathbb{P}_{1_\Gamma} = \widetilde{\mathbb{W}}_q(\rho),\ \mathbb{L}_{1_\Gamma} = \frac{2}{3}\widetilde{\mathbb{W}}_q(\rho);$$

$$\mathbb{P}_{3_\Gamma} = \frac{q-1-\widetilde{\mathbb{W}}_q(\rho)}{3},\quad \mathbb{L}_{3_\Gamma} = \frac{2(q-1-\widetilde{\mathbb{W}}_q(\rho))}{3},$$

$$\mathbb{P}_{0_\Gamma} = \mathbb{L}_{0_\Gamma} = \frac{2q-2\widetilde{\mathbb{W}}_q(\rho)+1}{3}.$$

The plane-line incidence matrix contains the same values of the point-line incidence matrix, but in this case they refer to Π_π, Λ_π instead of $\mathbb{P}_{\mathfrak{p}}, \mathbb{L}_{\mathfrak{p}}$.

Point-line and plane line matrix incidences for $\mathscr{L}_\rho$-lines

Theorem

Let q be odd. For the point-line incidence matrix corresponding to the orbit $\mathbb{O}_\rho$ of a line $\mathscr{L}_\rho$ the following holds:

- *Let $q \equiv -1 \pmod 3$. Then $\#\mathbb{O}_\rho = (q^3 - q)/2$ and*

$$\mathbb{P}_{\mathrm{T}} = 2,\ \mathbb{L}_{\mathrm{T}} = q - 1;\ \mathbb{P}_{1_\Gamma} = \mathbb{L}_{1_\Gamma} = \frac{q-1}{2};$$

$$6\mathbb{P}_{3_\Gamma} = 2\mathbb{L}_{3_\Gamma} = q - 5;\ 3\mathbb{P}_{0_\Gamma} = 2\mathbb{L}_{0_\Gamma} = q + 1.$$

Point-line and plane line matrix incidences for $\mathscr{L}_\rho$-lines

- Let $q \equiv 1 \pmod 3$.

$$\eta(\beta) = 1 \text{ if } \beta \in \square,\ \eta(\beta) = -1 \text{ if } \beta \in \not\square$$

$$\mathfrak{N}_{q,\rho} \triangleq \#\{\gamma \,|\, \gamma \in \mathbb{F}_q^*,\ \eta(1 + 4\rho^{-1}\gamma^3) = -1\}$$

Let -2ρ be a non-cube in $\mathbb{F}_q$. Then $\#\mathbb{O}_\rho = (q^3 - q)/3$;

$$\mathbb{P}_{\mathrm{T}} = 1,\ \mathbb{L}_{\mathrm{T}} = \frac{q-1}{3};\ \mathbb{P}_{1_\Gamma} = \mathfrak{N}_{q,\rho},\ \mathbb{L}_{1_\Gamma} = \frac{2}{3}\mathfrak{N}_{q,\rho};$$

$$\mathbb{P}_{3_\Gamma} = \frac{q - 1 - \mathfrak{N}_{q,\rho}}{3},\ \mathbb{L}_{3_\Gamma} = \frac{2(q - 1 - \mathfrak{N}_{q,\rho})}{3};$$

$$\mathbb{P}_{0_\Gamma} = \mathbb{L}_{0_\Gamma} = \frac{2q + 1 - 2\mathfrak{N}_{q,\rho}}{3}.$$

Point-line and plane line matrix incidences for $\mathscr{L}_\rho$-lines

- Let $q \equiv 1 \pmod 3$.

Let -2ρ be a cube in $\mathbb{F}_q$. Then $\#\mathbb{O}_\rho = (q^3 - q)/12$ and

$$\mathbb{P}_{\mathrm{T}} = 4,\ \mathbb{L}_{\mathrm{T}} = \frac{q-1}{3};\ \mathbb{P}_{1_\Gamma} = \mathfrak{N}_{q,\rho},\ \mathbb{L}_{1_\Gamma} = \frac{1}{6}\mathfrak{N}_{q,\rho};$$
$$\mathbb{P}_{3_\Gamma} = \frac{q-7-\mathfrak{N}_{q,\rho}}{3}, \mathbb{L}_{3_\Gamma} = \frac{q-7-\mathfrak{N}_{q,\rho}}{6};$$
$$\mathbb{P}_{0_\Gamma} = \frac{2(q-1-\mathfrak{N}_{q,\rho})}{3},\ \mathbb{L}_{0_\Gamma} = \frac{q-1-\mathfrak{N}_{q,\rho}}{6}.$$

The plane-line incidence matrix contains the same values of the point-line incidence matrix, but in this case they refer to Π_π, Λ_π *instead of* $\mathbb{P}_{\mathfrak{p}}, \mathbb{L}_{\mathfrak{p}}$

What about the orbits of lines of class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$?

Conjecture

Let q be odd, $q \equiv \xi \pmod 3$, $\xi \in \{1, -1, 0\}$.

The total number of $\mathrm{En}\Gamma$-line orbits is $2q - 3 + \xi$.

$n_q(\xi)$	*orbits of length*	$(q^3 - q)/4$,
$q - 1$	*orbits of length*	$(q^3 - q)/2$,
$(q - \xi)/3$	*orbits of length*	$q^3 - q$,

where

$$n_q(1) = (2q - 11)/3,$$
$$n_q(-1) = (2q - 10)/3,$$
$$n_q(0) = (2q - 6)/3.$$

In addition, for $q \equiv 1 \pmod 3$, *there are:*

1	*orbit of length*	$(q^3 - q)/12$,
2	*orbits of length*	$(q^3 - q)/3$.

What about the orbits of lines of class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$?

proved for odd $q, q \not\equiv 0 \pmod 3$ in:

K. Kaipa, N. Patanker, P. Pradhan,
On the $PGL_2(q)$-orbits of lines of $\mathrm{PG}(3,q)$ and binary quartic forms, arXiv:2312.07118 (2023)

The open problem of classifying binary quartic forms over $\mathbb{F}_q$ into G_q-orbits is solved and used

The Plücker embedding for the Klein quadric is applied

The incidence matrices are not considered and the stabilizer groups on language of $\mathrm{PG}(3,q)$ are not presented

What about the orbits of lines of class $\mathcal{O}_6 = \mathcal{O}_{\mathrm{En}\Gamma}$?

OPEN CASE

$q \equiv 0 \pmod 3$,

Find the $O(q)$ expected orbits that are not generated by lines of the family ℓ_μ

THANKS FOR ATTENTION!