Exceptional scattered polynomials in odd degree

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Linear square rank-metric codes

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$$\mathcal{L}_{n,q} = \frac{\left\{\sum_i a_i x^{q^i} \ :\ a_i \in \mathbb{F}_{q^n}\right\}}{\langle x^{q^n} - x\rangle}$$

$\Longrightarrow$ codes in $\mathcal{L}_{n,q}$ $\quad \mathrm{wt}(f) = \dim_{\mathbb{F}_q}(\mathrm{Im}(f)) \quad f : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$

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- 2-dim. $\mathbb{F}_{q^n}$-linear MRD codes:

$$\dim_{\mathbb{F}_q}(\mathrm{Ker}(f)) \leq 1 \quad \text{for all} \quad f \in C = \langle g(x), h(x) \rangle_{\mathbb{F}_{q^n}}$$

MRD codes and cattered polynomials

$C \leq_{q^n} \mathcal{L}(n,q)$: 2-dim. $\mathbb{F}_{q^n}$-linear MRD code

Up to equivalence:

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Many recent papers on scattered polynomials,

infinite families, also: q fixed, infinitely many n's

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$$f(x) = x^{q^\ell} \qquad \gcd(\ell, n) = 1$$

Scattered **binomials** of LP-type (Lunardon-Polverino 2001)

$$f(x) = \delta x^{q^{n-\ell}} + x^{q^\ell} \qquad \gcd(\ell, n) = 1 \quad \delta^{(q^n-1)/(q-1)} \neq 1$$

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$\langle x,\, \delta x^{q^{n-s}} + x^{q^s} \rangle_{\mathbb{F}_{q^n}} \quad \rightsquigarrow \quad$ right composition with invertible $x^{q^s} \in \mathcal{L}_{n,q}$

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$\langle C_\delta \rangle_{\mathbb{F}_{q^{nm}}} = \langle x^{q^\ell},\, \delta x + x^{q^{2\ell}} \rangle_{\mathbb{F}_{q^{nm}}} \subseteq \mathcal{L}_{nm,q} \quad$ is MRD iff $\quad \delta^{\frac{q^{nm}-1}{q-1}} \neq 1$

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$\Longrightarrow$ MRD over infinitely many extensions $\mathbb{F}_{q^{nm}}$ of $\mathbb{F}_{q^n}$

Exceptional scattered polynomials of index ℓ

$f(x) = \sum_{i=0}^{n-1} a_i x^{q^i} \in \mathbb{F}_{q^n}[x]$ **exceptional scattered** of index ℓ :

for infinitely many m : for all $y, z \in \mathbb{F}^*_{q^{nm}}$:

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others ?

ℓ-normalized exceptional scattered polynomials of index ℓ

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- $0 \leq \ell \leq n-1$, $q\text{-}\deg(f) \leq n-1$, $f(x)$ is monic
- the coefficient of x^{q^ℓ} in $f(x)$ is zero
- if $\ell > 0$, then the coefficient of x in $f(x)$ is nonzero

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Classification of exceptional scattered polynomials of index ℓ :

there are only monomials or LP-binomials, if

- $\ell \in \{0, 1, 2\}$: Bartoli-Zhou 2018, Bartoli-Montanucci 2021
- $\max\{\ell, q\text{-}\deg(f)\}$ is prime and q is odd : Ferraguti-Micheli 2021

Tools for exceptional scattered polynomials

Bartoli-Zhou 2018, Bartoli-Montanucci 2021: $(\ell \leq 2)$

$f(x)$ is scattered of index ℓ over $\mathbb{F}_{q^n}$ if and only if the curve

$\mathcal{C}: \; f(X)Y^{q^\ell} - f(Y)X^{q^\ell} = 0$ has no $\mathbb{F}_{q^n}$-rational affine points $(\bar{x}, \bar{y})$

out of the lines $X = 0, \; Y = \lambda X$ with $\lambda \in \mathbb{F}_q$

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Ferraguti-Micheli method $+$ group theory

Linear Galois groups

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S : splitting field of $F(x)$ over $\mathbb{F}_{q^n}(s)$

$m >> 0$ big enough (depending on $S : \mathbb{F}_{q^n}(s)$), $S_m = S \cdot \mathbb{F}_{q^{nm}}$

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$F(x)$ is separable and $\mathbb{F}_q$-linearized

$\Rightarrow$ $V = \{\text{roots of } F(x)\}$: d-dimensional $\mathbb{F}_q$-vector space

G_m^{geom}, G_m^{arith} act on V as subgroups of $\mathrm{GL}(V, \mathbb{F}_q)$

Scatteredness and linear Galois groups

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Theorem (Ferraguti-Micheli)

$f(x) \in \mathbb{F}_{q^{nm}}[x]$ is scattered of index ℓ $\iff$

for all $\alpha \in G_m^{\mathrm{geom}}$, for all $\gamma \in G_m^{\mathrm{arith}}$ with $\varphi_m(\gamma G_m^{\mathrm{geom}})|_{K_m} : x \mapsto x^{q^{nm}}$,

$$\mathrm{rank}_{\mathbb{F}_q}(\alpha\gamma - \mathrm{Id}) \geq d - 1$$

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Corollary

$f(x) \in \mathbb{F}_{q^{nm}}[x]$ scattered of index ℓ $\implies$ $G_m^{\mathrm{arith}} \neq G_m^{\mathrm{geom}}$

Transitive linear Galois groups

Galois groups: **transitive** on the roots of $F(x)/x = f(x)/x - sx^{q^\ell - 1}$

$\Rightarrow \quad G_m^{\mathrm{geom}}, G_m^{\mathrm{arith}} \leq \mathrm{GL}(d,q)$ are **transitive** on $\mathbb{F}_q^d \setminus \{0\}$

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$$\mathrm{GL}(d, q = p^h) \leq \mathrm{GL}(dh, p)$$

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$$\Downarrow$$

Ferraguti-Micheli :

if d is prime and q is odd $\quad \Rightarrow \quad (x^{q^\ell}, f(x)) = (x^{q^d}, x) \quad$ (monomial case)

$$G_m^{\mathrm{geom}} \cong \mathrm{GL}(1, q^d) \qquad G_m^{\mathrm{arith}} \cong \Gamma\mathrm{L}_q(1, q^d)$$

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- $\mathrm{SL}(e, q^{d/e}) \triangleleft G \leq \Gamma\mathrm{L}_q(e, q^{d/e})$ for some divisor $1 \leq e \mid d$
- $\mathrm{Sp}(e, q^{d/e}) \triangleleft G \leq \Gamma\mathrm{L}_q(e, q^{d/e})$ for some even divisor $4 \leq e \mid d$
- $G_2(2^{d/6})' \triangleleft G \leq \Gamma\mathrm{L}_q(6, 2^{d/6})$ with q even and $6 \mid d$
- sporadic groups with $q^d \in \{5^2, 7^2, 11^2, 23^2, 29^2, 59^2, 2^4, 3^4, 3^6\}$

here: $\Gamma\mathrm{L}_q(a, q^b) = \mathrm{GL}(a, q^b) \rtimes \mathrm{Aut}(\mathbb{F}_{q^b} : \mathbb{F}_q)$

Hering + Ferraguti-Micheli + linear groups

$f(x) \in \mathbb{F}_{q^n}[x]$ ℓ-normalized $\quad d = \max\{\ell, q\text{-}\deg(f)\}$ $\quad m >> 0$

$f(x)$ exceptional scattered of index ℓ, then:

- $\mathrm{SL}(e, q^{d/e}) \triangleleft G_m^{\mathrm{geom}} \triangleleft G_m^{\mathrm{arith}} \leq \Gamma\mathrm{L}_q(e, q^{d/e})$ for some $1 \leq e \mid d$
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We can exclude a case when:

for any $\gamma \in G_m^{\mathrm{arith}}$, we find $\alpha \in G_m^{\mathrm{geom}}$ such that $\mathrm{rank}_{\mathbb{F}_q}(\alpha\gamma - \mathrm{Id}) < d - 1$

$\Rightarrow$ contradiction to Ferraguti-Micheli characterization of scatteredness!

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Closer analysis of:

- natural embeddings $\Gamma\mathrm{L}_q(e, q^{d/e}) \hookrightarrow \mathrm{GL}(d, q)$
- Linear groups normalizing a group of type SL or Sp

Restrictions on G_m^{geom} and G_m^{arith}

We can exclude the cases $G \triangleleft G_m^{\mathrm{geom}} \triangleleft G_m^{\mathrm{arith}}$ with

- $G = \mathrm{SL}(e, q^{d/e})$ for any $e \geq 3$
- $G = \mathrm{Sp}(e, q^{d/e})$ for any $e \geq 4$

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$\Rightarrow$ monomial case, $G_m^{\mathrm{geom}} = \mathrm{GL}(1, q^d)$, $G_m^{\mathrm{arith}} = \Gamma\mathrm{L}_q(1, q^d)$

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Theorem (Giulietti-Z.)

q any prime power, $f(x) \in \mathbb{F}_{q^n}[x]$,

$f(x)$ *exceptional scattered of index* ℓ $+$ $\max\{\ell, q\text{-}\deg(f)\}$ *odd*

$$\Downarrow$$

$f(x)$ *is a monomial of pseudoregulus type*

Restrictions on G_m^{geom} and G_m^{arith}, even case

$f(x) \in \mathbb{F}_{q^n}[x]$ exceptional scattered of index ℓ

$f(x)$ NOT a monomial $\quad d = \max\{\ell, q\text{-}\deg(f)\}$ **even**

Restrictions on G_m^{geom} and G_m^{arith}, even case

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2. $G_2(2^{d/6})' \lhd G^{\mathrm{geom}} \lhd G^{\mathrm{arith}} \leq \Gamma\mathrm{L}_q(6, 2^{d/6})$ with q even and $6 \mid d$
3. $q^d \in \{5^2, 7^2, 11^2, 23^2, 29^2, 59^2, 2^4, 3^4, 3^6\}$

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Example

LP- binomial $f(x) = x^{q^{2\ell}} + \delta x$ over $\mathbb{F}_{q^n}$ $\quad \gcd(\ell, n) = 1 \quad \delta^{\frac{q^n-1}{q-1}} \neq 1$

excetpional scattered of index ℓ

$G^{\mathrm{geom}} \cong \mathrm{SL}(2, q^{\ell}) \qquad G^{\mathrm{arith}} \cong \Sigma\mathrm{L}_q(2, q^{\ell}) = \mathrm{SL}(2, q^{\ell}) \rtimes \mathrm{Aut}(\mathbb{F}_{q^{\ell}} : \mathbb{F}_q)$

Thank you for your attention!